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LETTER TO THE EDITOR

The statistics of light scattered by a random phase screen

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Abstract. We report some preliminary results of an investigation of the statistical properties of light scattered by a deep random phase screen, with particular reference to the nongaussian regime when the dimensions of the illuminated area are of the same order of magnitude as the phase correlation length.

In the majority of light scattering experiments to date the spatial extent of the refractive index inhomogeneities giving rise to the scattering has been very much smaller than the region of sample illuminated by the laser light source. The instantaneous electric field at the detector is then the sum of very many randomly phased components, and therefore the single interval statistics of this field closely approximates gaussian (see for example Jakeman *et al* 1968). When the number of independent scattering centres is small, however, marked departures from gaussian statistics occur (see for example Bourke *et al* 1970, Schaefer and Pusey 1972, Bluemel *et al* 1972). Such departures are characteristic of the sample and of the optical arrangement so that the statistics in this case carry more information than in the gaussian limit.

In this letter we investigate a light scattering system in many examples of which the nongaussian regime is accessible for experimental investigation: namely the deep, random phase screen. Such a system simply retards the phase of an incident electromagnetic field by a randomly varying, position-dependent amount typically equivalent to many wavelengths path difference. Familiar phenomena caused in this way are the twinkling of starlight and the swimming pool effect (Taylor 1972). Examples of deep phase screens of considerable current interest are moving diffuse surfaces such as ground glass (see for example Estes *et al* 1971) and the dynamic scattering mode exhibited by thin layers of nematic liquid crystals (Deutsch and Keating 1969). A good deal of literature exists on the diffuse surface problem particularly, but as far as we are aware no theoretical analysis of the statistics of the scattered light has been carried out for the nongaussian case. We shall outline here two approaches to this problem based on the assumption of joint-gaussian statistics for the phase $\phi(\boldsymbol{r})$ of the light emerging from a point $\boldsymbol{r}$ of the planar source region (figure 1). We shall assume the scatterer to be illuminated by a focused down laser beam of width W_0.

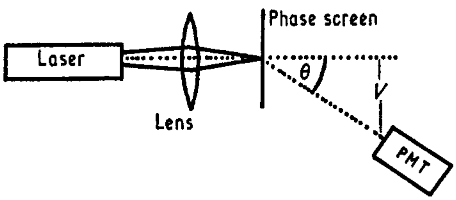

Figure 1. Schematic diagram of optical arrangement.

Following first a continuum approach (see for example Mercier 1962) let the positive frequency part of the electric field emerging from the scattering region be given by

$$\mathscr{E}^{+}(\boldsymbol{r};t) = E_0 \exp\{\mathrm{i}(\phi(\boldsymbol{r})-\omega t)-r^2/W_0{}^2\}. \tag{1}$$

With the help of the Helmholtz formula, the intensity in front of the phase screen may be written down in a far-field approximation as follows:

$$\langle I(\theta)\rangle = \langle \mathscr{E}^{+}(\theta;t)\mathscr{E}^{-}(\theta;t)\rangle = |E_0|^2 \int_{-\infty}^{+\infty} \mathrm{d}^2r'\mathrm{d}^2r'' \, \langle \exp \mathrm{i}(\phi(\boldsymbol{r}'-\boldsymbol{r}'')-\phi(\boldsymbol{r}'))\rangle$$
$$\times \exp[\mathrm{i}k|\boldsymbol{r}''|\sin\theta\cos\eta - \{(\boldsymbol{r}'-\boldsymbol{r}'')^2+\boldsymbol{r}'^2\}/W_0{}^2]$$

where θ is defined in figure 1, η is the angle between $\boldsymbol{r}''$ and the radius vector $\boldsymbol{V}$, and $\boldsymbol{k}$ is the wavevector of the light. The average in the integrand may be evaluated using the joint-gaussian hypothesis for ϕ:

$$\langle \exp \mathrm{i}(\phi(\boldsymbol{r}'-\boldsymbol{r}'')-\phi(\boldsymbol{r}'))\rangle = \exp\{-\overline{\phi^2}(1-\langle\phi(\boldsymbol{r}'-\boldsymbol{r}'')\phi(\boldsymbol{r}')\rangle/\overline{\phi^2})\} = \exp\{-\overline{\phi^2}(1-\rho(r''))\} \tag{2}$$

where $\overline{\phi^2}$ is the mean-square phase deviation and $\rho(r)$ is the normalized phase autocorrelation function. After integration we obtain the result

$$\langle I(\theta)\rangle = \pi^2 W_0{}^2|E_0|^2 \int_0^{\infty} r\mathrm{d}r \, J_0(kr\sin\theta)\exp\{-\overline{\phi^2}(1-\rho(r))-r^2/2W_0{}^2\}. \tag{3}$$

A formula for the second moment of the intensity fluctuation distribution,

$$\langle I^2(\theta)\rangle = \pi W_0{}^2|E_0|^4 \int_{-\infty}^{+\infty} \mathrm{d}^2r'\mathrm{d}^2r''\mathrm{d}^2r''' \exp\{2\mathrm{i}kr''\sin\theta\cos\eta-(r'^2+r''^2+r'''^2)/W_0{}^2\}$$
$$\times \exp\{-\overline{\phi^2}(2-\rho(r''+r''')-\rho(r''-r''')-\rho(r'+r'')-\rho(r'-r'')$$
$$+\rho(r'+r''')+\rho(r'-r'''))\}, \tag{4}$$

may be derived similarly. We have evaluated equations (3) and (4) in the limit $\overline{\phi^2} \gg 1$ by a method of steepest descents, assuming only that the function ρ has a curvature

$$\left.\frac{\mathrm{d}^2\rho(r)}{\mathrm{d}r^2}\right|_{r=0} = -\frac{2}{\xi^2} \tag{5}$$

at the origin where $\xi \ll W_0(\overline{\phi^2})^{1/2}$ may be interpreted as a characteristic phase correlation length (Marathay *et al* 1970). The formulae obtained in this way, namely

$$\langle I\rangle = \frac{\pi^2 W_0{}^2\xi^2|E_0|^2}{2\overline{\phi^2}}\exp\left(\frac{-k^2\xi^2\sin^2\theta}{4\overline{\phi^2}}\right) \tag{6}$$

$$\frac{\langle I^2\rangle}{\langle I\rangle^2} = 2-\frac{2\xi^2}{W_0{}^2}+\frac{\xi^2\overline{\phi^2}}{4W_0{}^2}\exp\left(\frac{k^2\xi^2\sin^2\theta}{4\overline{\phi^2}}\right) \tag{7}$$

should provide an adequate description of the low order statistics of light scattered from a deep random phase screen.

In the limit $\xi \ll W_0$ the second moment (7) assumes the gaussian value of two as expected but as W_0 is reduced the presence of the large $\overline{\phi^2}$ factor leads to a prediction of values in excess of two, particularly at large angles where the effect of the exponential term may become significant. The size of this factor is closely related to the angular distribution of intensity (6) so that its importance can be estimated without recourse to small beamwidths.

Experiments in which light was scattered from thin layers of nematic liquid crystal undergoing electrohydrodynamic turbulence (Jakeman and Pusey 1973) have confirmed both the angle and beamwidth dependencies predicted by (6) and (7). The values of $\overline{\phi^2}$ and ξ were estimated using these formulae to be of the order of 36 and 2 μm respectively for this system so that the nongaussian regime is easily accessible by focusing down the incident laser beam. Under such conditions a 'lighthouse' effect can be observed (sometimes with the naked eye) in which the incident radiation is scattered into a few well defined directions which fluctuate randomly with time.

Extension of the continuum approach to higher moments becomes progressively more difficult. However, a 'micro-area' model used by Enloe (1967) and more recently by Estes *et al* (1971) enables more general results to be obtained within the framework of the above approximations. We imagine the illuminated area to be made up of N regions R giving statistically independent contributions to the far field. This is equivalent to neglecting terms like $\exp(-\overline{\phi^2})$ in the earlier method. Thus we may write

$$\mathscr{E}^+(\theta;t) = \sum_{j=1}^{N} a_j(\theta;t)\exp(\mathrm{i}\psi_j) \tag{8}$$

where the ψ_j are statistically independent random phases, whilst the real diffraction factors are given by

$$a_j^2(\theta;t) = |E_0|^2 \int_R \int_R \exp\{\mathrm{i}(k|\boldsymbol{r}-\boldsymbol{r}'|\sin\theta\cos\eta + \phi_j(\boldsymbol{r}) - \phi_j(\boldsymbol{r}'))\}\mathrm{d}^2r\mathrm{d}^2r' \tag{9}$$

where η is the angle between $\boldsymbol{r}-\boldsymbol{r}'$ and the radius vector to the detector. Expression (8) describes a finite random walk in the complex $\mathscr{E}^+$ plane with variable step length. This problem was analysed by several authors including Lord Rayleigh (1919) and the relevant distribution is quoted by Watson (1944). The results have been applied recently in the special case of constant step length by Pusey *et al* (1973) to evaluate the properties of light scattered from a finite number of independent particles. The generating function of the distribution of intensity for a fixed set of diffraction factors is given by

$$\langle\exp(-\lambda I)\rangle = \Psi_2(1;1,1,1,\ldots 1; -a_1^2\lambda, -a_2^2\lambda \ldots -a_N^2\lambda) \tag{10}$$

where Ψ_2 is a confluent hypergeometric function of N variables. Assuming that the fluctuations of all the a_j can be described by a single distribution function, (10) leads to the following expressions for the first few moments:

$$\langle I\rangle = N\langle a^2\rangle \tag{11a}$$

$$\langle I^2\rangle = N\langle a^4\rangle + 2N(N-1)\langle a^2\rangle^2 \tag{11b}$$

$$\langle I^3\rangle = N\langle a^6\rangle + 9N(N-1)\langle a^2\rangle\langle a^4\rangle + 6N(N-1)(N-2)\langle a^2\rangle^3 \tag{11c}$$

and higher moments can be generated without difficulty. The moments of a^2 can be evaluated from (9) using the joint-gaussian hypothesis for the statistics of $\phi_j(\boldsymbol{r})$ and an approximation similar to (5), taking advantage of the fact that R is characterized by the dimension ξ. A more intuitive but mathematically equivalent technique is to assume that $\phi_j(\boldsymbol{r})$ is coherent and varies linearly with distance within R, the gradient being gaussian distributed. In either case it is convenient to replace the integrals over R in (9) by infinite integrals over the gaussian weighting function $\exp(-r^2/\xi^2)$. We then obtain

$$\langle a^{2n}\rangle = \frac{2(\pi\xi^2|E_0|/2)^{2n}}{n\overline{\dot{\phi}^2}}\exp\left(\frac{-R^2\xi^2\sin^2\theta}{4\overline{\dot{\phi}^2}}\right). \tag{12}$$

Substitution of (12) into (11*a*) and (11*b*) immediately recovers the results (6) and (7) when N is interpreted as the number of phase coherence areas per illuminated area, $W_0{}^2/\xi^2$.

It is not difficult to show from (10) and (12) that the higher normalized moments $n^{(r)} = \langle I^r\rangle/\langle I\rangle^r$ may be expressed entirely in terms of N and $n^{(2)}$. This 'factorization' property is illustrated in figure 2 by the full curves. We have taken $N = 16$ to make

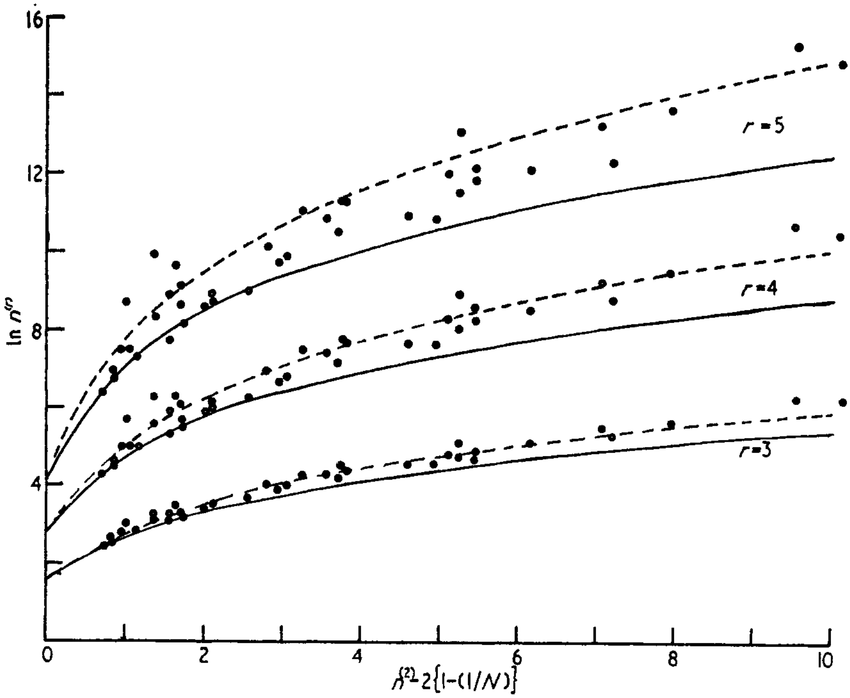

Figure 2. Higher-order statistical properties: comparison of experiment with theory. Full curve, phase fluctuations only; broken curve, phase and amplitude fluctuations.

contact with experiment but the curves do not change significantly, for fixed $n^{(2)}$, as $N\to\infty$. Also shown in figure 2 are some experimental results for the liquid crystal system referred to earlier (Jakeman and Pusey 1973). The discrepancy between experiment and theory which is evident at high values of $n^{(2)}$, particularly in the case of the high moments, is probably due to amplitude modulation associated with depolarization of the incident light. This may be taken into account in a simple way by multiplying equation (9) by an attenuation factor $\exp(-\sigma)$ which is uniform over each micro-area. The assumption of gaussian statistics for σ then leads to an additional 'log-normal' factor $\exp\{n(n-1)\overline{\sigma^2}/2\}$ on the right-hand side of (12). The broken curves in figure 2 are obtained by taking $\exp(\overline{\sigma^2}) = 2$ and show good agreement with the experimental data.

It is clear from our preliminary results that nongaussian statistics can be a valuable source of information in light scattering systems such as the deep random phase screen for which an adequate theoretical description can be found. In addition to the examples we have already quoted, the phase screen model described in this letter should apply to a thin film of pure fluid or binary liquid mixture sufficiently close to the critical point for the range of correlation of the fluctuations to be comparable with the size of the illuminated region, provided that the associated refractive index changes are large enough. In this connection both the time dependence of the nongaussian intensity fluctuations and the spatial cross-correlation functions obtained by multiple-detector experiments (Cantrell 1968, Swift 1973) are plainly of great interest. The weak phase-screen situation which may obtain for smaller refractive index changes in critical systems remains to be investigated.

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